Kent Denver Innovation Scholars Program
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Technology and Design Track
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Computational Physics

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OVERVIEW

I set out to pursue a project in computational physics in order to gain a deeper understanding of the intersection between Math, Physics, and Computer Science. My goal was to simulate and study a complex chaotic dynamical system for the purpose of using mathematical techniques to analyze chaos and acquiring the skills to write simulations for real-world applications.

RESEARCH

Research for this project primarily consisted of the selection of a physical system as well as the study of techniques and procedures necessary to run such a simulation.

I chose to simulate a "pendulum-cart," a system which as far as I know has never before been directly simulated or researched anywhere else. This chosen system is pictured to the right: x and θ are able to move freely under the influence of gravity and f(x) can be any continuous, twice differentiable function.

The researched techniques include: Fourth order Runge-Kutta, Lagrangian mechanics, Euler-Lagrange equation, and basic chaos theory.

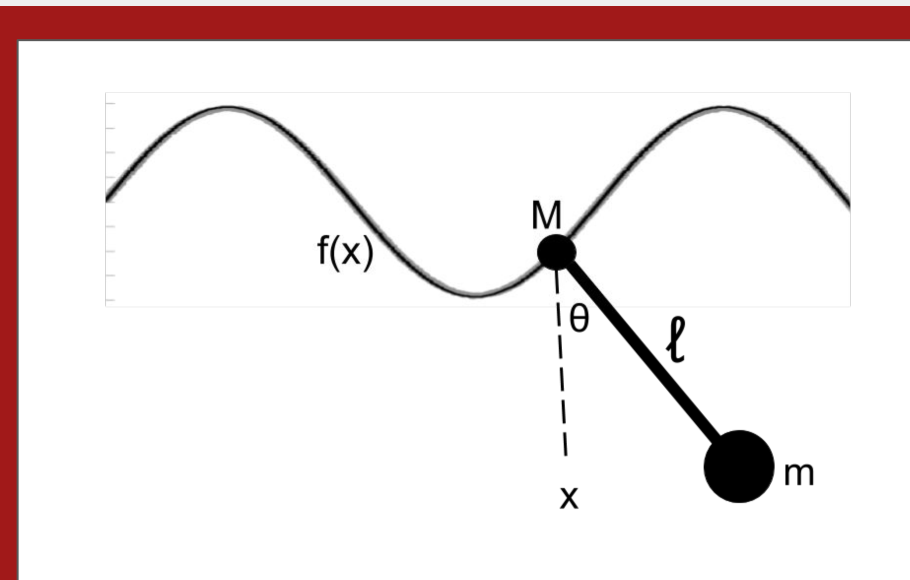

FIGURE 1: Diagram of the selected chaotic system

PROCEDURE

The procedure for this project consisted of two steps: first, mathematically modelling and simulating the system using java, then performing analysis on regions of non-chaos.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$

FIGURE 2: Formula used to derive equations of motion (Euler-Lagrange equation)

An overview of the steps taken is as follows:

1. Derive equations of motion through Lagrangian mechanics
2. Write a class describing the state of the system and its time derivatives
3. Write an integrator using fourth-order Runge-Kutta (RK4) to run the system
4. Analyze energy to verify accuracy of the equation of motion
5. Write an algorithm to quantitatively measure chaos for a given initial state
6. Test thousands of initial states to develop an understanding of which regions of phase space are chaotic
7. Use mathematical analysis to explain why those regions are chaotic

ANALYSIS

After finalizing a diagram describing regions of phase space in which the system is chaotic, it appeared to be completely chaotic for the majority of initial setups. However, there were three distinct non-contiguous regions of non-chaos: a central ellipse, a line passing through the origin with a slope of -2, and two distinct "islands" around *xDot0=0* and *thetaDot0=15*.

1. **The central ellipse**

This region of non-chaos had perhaps the most straightforward explanation. Obviously, the region right around the origin would be non-chaotic (since it would have no kinetic energy). However, in order to further verify the fact that the ellipse was due to a lack of kinetic energy, I graphed lines of constant energy in phase space which lined up exactly with the ellipse, showing this was the reason for non-chaos in that area.

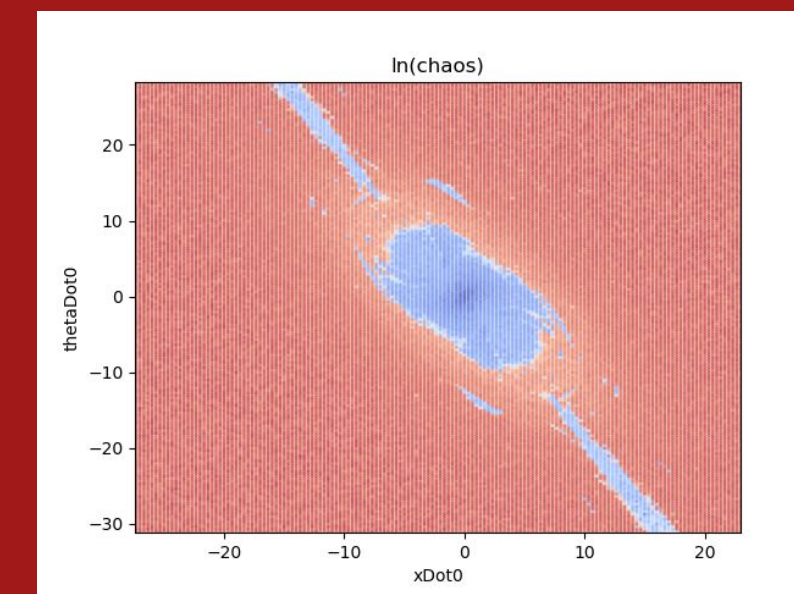

FIGURE 3: An analysis of how chaotic the system is as a function of initial velocity and angular velocity

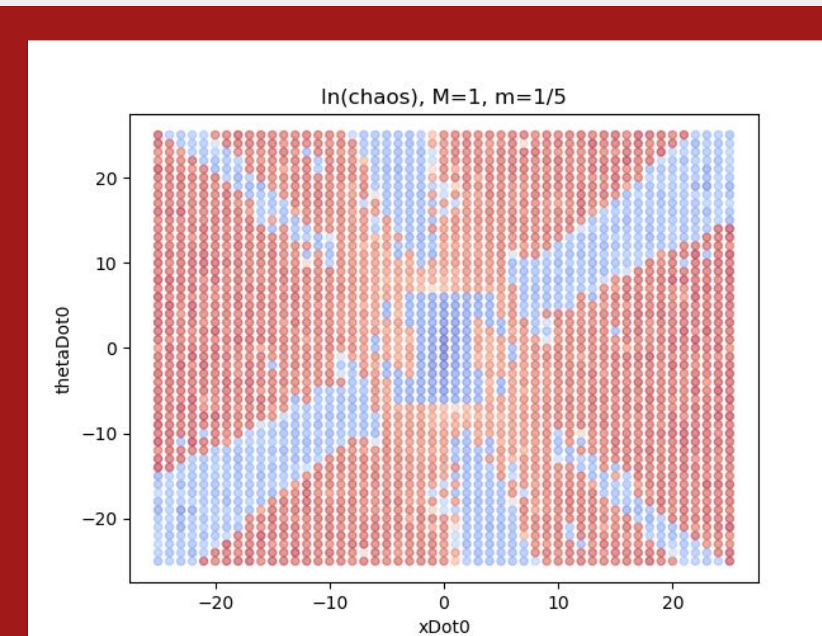

FIGURE 4: The line of non-chaos becomes steeper with a greater mass ratio

2. The two "islands"

While the cause of this region of non-chaos is not readily apparent from the diagram, watching a simulation from this region of phase space makes the source obvious. When *thetaDot0* is around 15 [rad/sec], the frequency of the pendulum lines up perfectly with the frequency of the sine wave, causing the system to slip into a consistent non-chaotic pattern as it rotates one revolution for every period of the sine wave it passes over.

3. The diagonal line

This region was the most difficult to study, as its cause was not apparent from the chaos diagram nor a run of the simulation. However, after running several iterations to find that the slope of the line correlated strongly to the ratio of the masses, it was discovered that this line appears where initial linear momentum is zero. Meaning that despite having high kinetic energy, the system is not able to move any significant distance and is therefore non-chaotic.

RESULTS/CONCLUSION

After graphing the system's chaos in phase space and performing analysis, it became clear that I had chosen a unique system in which there were regions of both chaos and non-chaos explained by several different physics phenomena. Unfortunately, I was not able to find an application for this simulation in the real world as it is highly idealized, neglecting forces such as elasticity and friction. However, the code I wrote provides a solid framework to further study related physical systems.